Section 4.4 Compound Angles

A trigonometric expression that depends on two or more angles is known as a compound angle expression.

Trig ratio's do not have a distributive property –

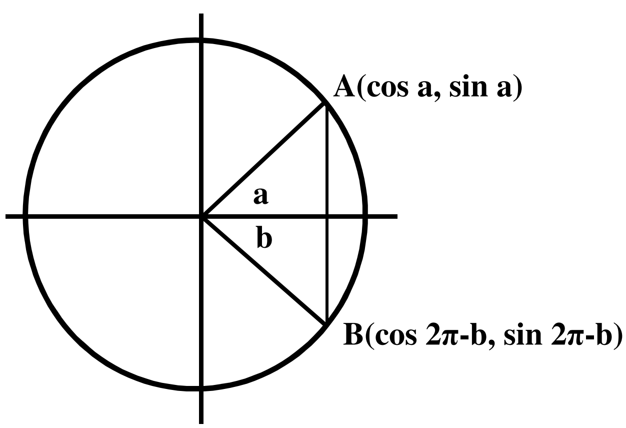
Ex. $2(x + 1) = 2x + 2$

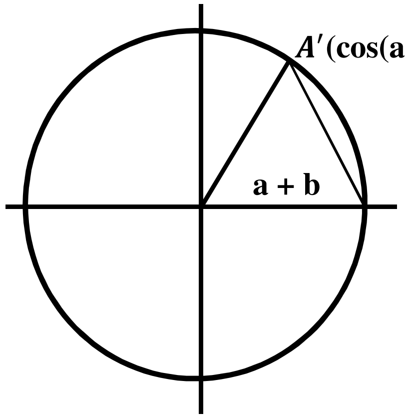
You can't just say that $\sin(x + 1) = \sin x + \sin 1$ – pick any angle combination and test for yourself, you will see that it simply does not work.

For trig ratio's expansion of double angles is a little more complex.

Let's start with the **Addition Formula for Cosine:**

Picture a unit circle with two angles, **a** and **b** which connect the center of the circle to points **A**, **B** respectively as shown below. We could connect these two points with secant $\overline{AB}$.





Let's rotate the whole thing by angle **b,** that way we have a nicer starting point, an ending point that has coordinates with the relation we want to find [cos(a + b)] and a secant distance $\overline{A'B'}$ that has to be exactly the same distance as secant $\overline{AB}$.

Pythagorean Theorem Time – let us calculate the length of both secants and set them equal to each other.

$\overline{AB} = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$ **A(cos a, sin a) , B(cos 2π-b, sin 2π-b)**

Before we start we are going to make use of a couple of the identities we created in the last section. Specifically $cos(2\pi - x) = \cos x$ and $sin(2\pi - x) = -\sin x$ to be used in point B, you can thank me later ☺.

$\overline{AB} = \sqrt{(-\sin b - \sin a)^2 + (\cos b - \cos a)^2}$

$\overline{A'B'} = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$ ***A′*****(cos(a + b), sin(a + b)) ,** ***B′*****(1,0)**

$\overline{A'B'} = \sqrt{(0 - \sin(a + b))^2 + (1 - \cos(a + b))^2}$

Now let us set the equal and expand.

$$\sqrt{(-\sin b - \sin a)^2 + (\cos b - \cos a)^2} = \sqrt{(1 - \sin(a + b))^2 + (0 - \cos(a + b))^2}$$

$$(-\sin b - \sin a)^2 + (\cos b - \cos a)^2 = (1 - \sin(a + b))^2 + (0 - \cos(a + b))^2$$

$$\sin^2 b + 2\sin a \sin b + \sin^2 a + \cos^2 b - 2\cos a \cos b + \cos^2 a$$
$$= \sin^2(a + b) + 1 - 2\cos(a + b) + \cos^2(a + b)$$

Recall $\sin^2(x) + \cos^2(x) = 1$ Pythagorean Identity

So

$\underline{\sin^2 b} + 2\sin a \sin b + \underline{\sin^2 a} + \underline{\cos^2 b} - 2\cos a \cos b + \underline{\cos^2 a} = \underline{\sin^2(a+b)} + 1 - 2\cos(a+b) + \underline{\cos^2(a+b)}$

Becomes

$1 + 1 + 2\sin a \sin b - 2\cos a \cos b = 1 + 1 - 2\cos(a+b)$

$2\sin a \sin b - 2\cos a \cos b = -2\cos(a+b)$

$-\sin a \sin b + \cos a \cos b = \cos(a+b)$

Or $\boldsymbol{\cos(a+b) = \cos a \cos b - \sin a \sin b}$

Subtraction Formula for Cosine (we will take the lazy approach)

If $\boldsymbol{\cos(a+b) = \cos a \cos b - \sin a \sin b}$, and we want cos (a – b), let's just sub in a negative b value into the formula.

$\boldsymbol{\cos(a + (-b)) = \cos a \cos(-b) - \sin a \sin(-b)}$

$\boldsymbol{\cos(a - b) = \cos a \cos(2\pi - b) - \sin a \sin(2\pi - b)}$

$\boldsymbol{\cos(a - b) = \cos a \cos(2\pi - b) - \sin a \sin(2\pi - b)}$

$\boldsymbol{\cos(a - b) = \cos a \cos b - \sin a\, (-\sin\ b)}$

$\boldsymbol{\cos(a - b) = \cos a \cos b + \sin a \sin b}$

Addition Formula for Sine

Recall

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

$$\cos(-x) = \cos x$$

$$\sin(-x) = -\sin x$$

So

$$\sin\,(\mathrm{a} + \mathrm{b}) = \cos\left(\frac{\pi}{2} - (a + b)\right)$$

$$\sin\,(\mathrm{a} + \mathrm{b}) = \cos\left((\frac{\pi}{2} - a) - b\right)$$

$$\boldsymbol{sin}(\boldsymbol{a} + \boldsymbol{b}) = \mathbf{cos}(\frac{\pi}{2} - a)\,\mathbf{cos}\,\boldsymbol{b} + \mathbf{sin}(\frac{\pi}{2} - a)\,\mathbf{sin}\,\boldsymbol{b}$$

$$\boxed{\boldsymbol{sin}(\boldsymbol{a} + \boldsymbol{b}) = \mathbf{sin}\,\boldsymbol{a}\,\mathbf{cos}\,\boldsymbol{b} + \mathbf{cos}\,\boldsymbol{a}\,\mathbf{sin}\,\boldsymbol{b}}$$

Subtraction Formula for Sine

Let b = - b as we did for cosine - yours to finish.

Example: Show that the formula $cos(x - y) = cosxcosy + sinxsiny$ is true for $x = \pi/6$ and $y = \pi/3$

Example: Show that the formula $sin(x + y) = sinxcosy + cosxsiny$ is true for $x = \pi/2$ and $y = 3\pi/4$

Examples: Use an appropriate compound angle formula to express the following as a single trigonometric function and then determine an exact value for each.

a) $\sin\frac{\pi}{4}\cos\frac{\pi}{12}-\cos\frac{\pi}{4}\sin\frac{\pi}{12}$

b) $\cos\frac{10\pi}{9}\cos\frac{5\pi}{18}+\sin\frac{10\pi}{9}\sin\frac{5\pi}{18}$

Example: Use an appropriate compound formula to determine an exact value for $\cos\frac{5\pi}{12}$

Example: Angles x and y are located in the first quadrant such that sin x $= \frac{5}{13}$ and $\cos y = \frac{3}{5}$
Determine an exact value for $sin(x - y)$.

Example: Use an appropriate compound angle formula to

a) Prove the double angle formula for sine, $sin2\theta = 2sin\theta cos\theta$.

b) Prove the double angle formula for cosine, $cos2\theta = cos^2\theta - sin^2\theta$
or $= 2cos^2\theta - 1$
or $= 1 - 2sin^2\theta$