Section 3.5 Making Connections with Rational Functions and Equations

When solving a problem, it's important to read carefully to determine whether a function is being analyzed (Finding key features) or an equation or inequality is to be solved (find a missing value).

A full analysis will involve four components:

1. Numeric (tables, ordered pairs, calculations)
2. Algebraic (formulas, solving equations)
3. Graphical
4. Verbal (descriptions)

When investigating special cases of functions, factor and reduce where possible. Indicate the restrictions on the variables in order to identify hidden discontinuities.

When investigating new types of rational functions, consider what is different about the coefficients and the degree of the polynomials in the numerator and denominator. These differences could affect the stretch factor of the curve and the equations of the asymptotes and they could cause other discontinuities.

Discontinuities are values at which a function becomes undefined. They may appear as asymptotes or as holes (or gaps).

Summary of Rational Functions:

- The quotient of two polynomial functions results in a rational function which often has one or more **discontinuities**.

- The breaks or discontinuities in a rational function occur when the function is undefined. The function is undefined at values where the denominator equals zero. As a result, these values must be restricted from the domain of the function.

- The values that must be restricted from the domain of a rational function results in key characteristics that define the shape of the graph. These characteristics include a combination of vertical asymptotes (also called infinite discontinuities) and holes (also called point discontinuities).

- The end behaviours of many rational functions are determined by either horizontal asymptotes or oblique asymptotes.

* A rational function $f(x) = \frac{P(x)}{Q(x)}$ has a **hole** at **x = a** if $\frac{P(a)}{Q(a)} = \frac{0}{0}$. This occurs when P(x) and Q(x) contain a **common factor** of **(x-a)**.

For example: $f(x) = \frac{x^2-4}{x-2}$ has the common factor of (x-2) in the numerator and the denominator. This results in a hole in the graph of f(x) at x = 2

* A rational function $f(x) = \frac{P(x)}{Q(x)}$ has a **vertical asymptote** at **x = a** if $\frac{P(a)}{Q(a)} = \frac{P(a)}{0}$.

  For example: $f(x) = \frac{x+1}{x-2}$ has a vertical asymptote at x = 2

* A rational function $f(x) = \frac{P(x)}{Q(x)}$ has a **horizontal asymptote** only when the **degree of P(x) is less than or equal to the degree of Q(x)**. The equation of the horizontal asymptote is the ratio of the leading coefficients in the numerator and denominator.

  For example: : $f(x) = \frac{2x}{x+1}$ has a horizontal asymptote at y = 2.

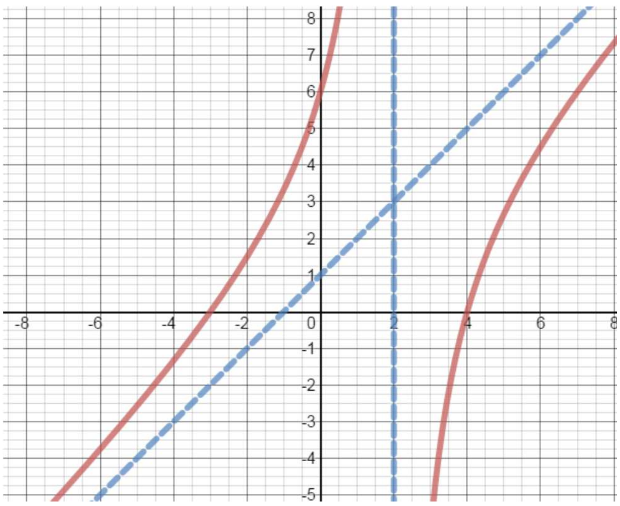
  Therefore, any reciprocal linear or reciprocal quadratic function has a horizontal asymptote at y = 0 because the degree of the numerator is 0, and the degree of the denominator is either 1 or 2.

* A rational function $f(x) = \frac{P(x)}{Q(x)}$ has an **oblique (slant) asymptote** only when the **degree of P(x) is greater than the degree of Q(x) by exactly 1**.

  To determine the equation of an oblique asymptote, divide the numerator, P(x), by the denominator, Q(x), using long division. The dividend is the equation of the oblique asymptote.

  For example, $f(x) = \frac{x^2-x-2}{x-2}$ has an oblique asymptote defined by y = x + 1

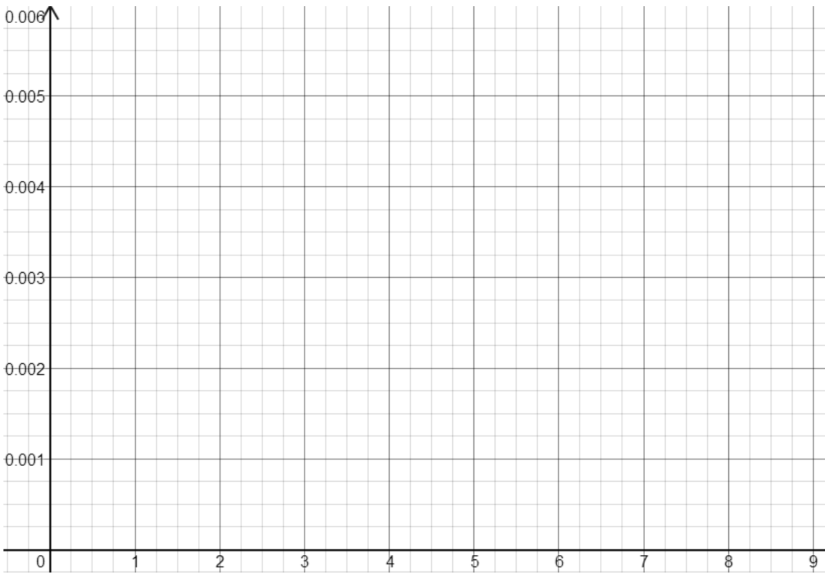
$$\begin{array}{r|l} & x+1 \\ \hline x-2 & x^2-x-2 \\ & x^2-2x \\ \hline & \quad x-2 \\ & \quad x-2 \\ \hline & \qquad 0 \end{array}$$



Example: The intensity of sound, in watts per square meter, varies inversely as the square of the distance, in meters, from the source of the sound. So, where k is a constant. The intensity of the sound from a loudspeaker at a distance of 2 m is 0.001 W/m^2.

a) Determine the value of k, and then determine the equation of the function to represent this relationship. Write an appropriate restriction on the variable.

b) Graph the function.

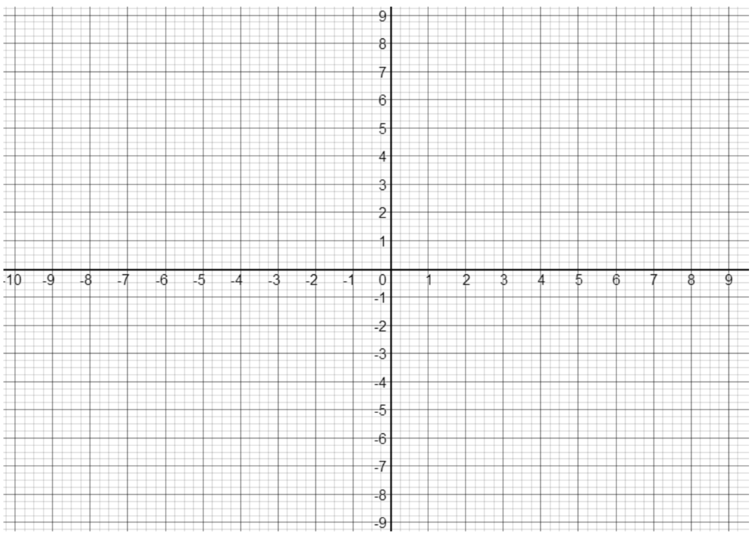


c) What is the effect of halving the distance from the source of the sound?

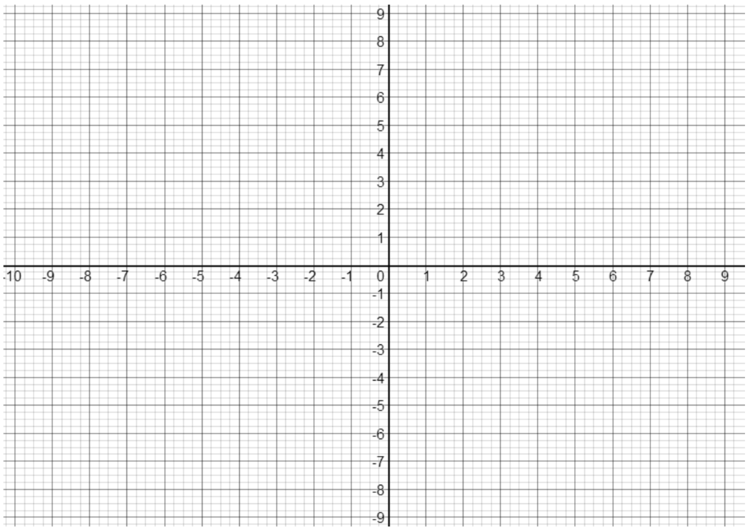
Example: In order to create a saline solution, salt water with a concentration of 40 g/L is added at a rate of 500 L/min to a tank of water that initially contained 8000 L of pure water. The resulting concentration of the solution in the tank can be modeled by the function $C(t) = \frac{40t}{160+t}$ where C is the concentration, in grams per litre, and t is the time, in minutes. In how many minutes will the saline concentration be 30 g/L?

Example: Simplify the following rational functions and state restrictions on the variable. Then sketch a graph of each simplified function and explain how these are special cases.

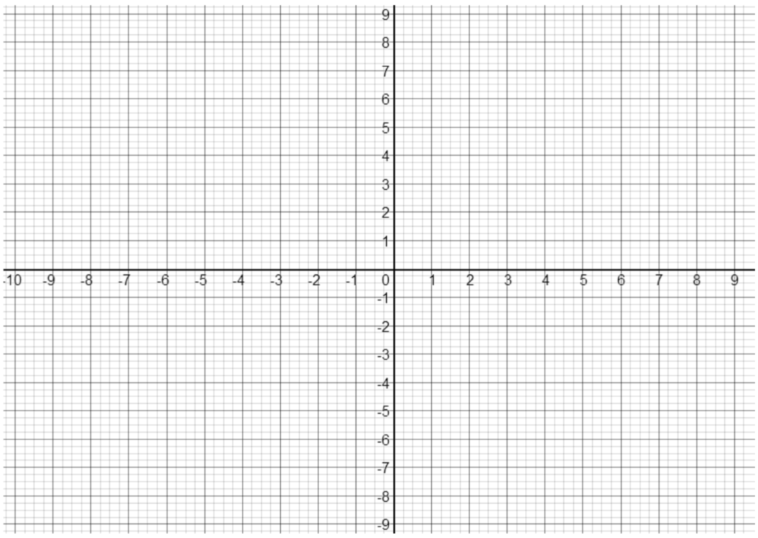
a) $f(x) = \frac{3x^2-8x+5}{x-1}$



b) $f(x) = \frac{2x^2+x-3}{2x^2+7x+6}$



c) $f(x) = \frac{2x^2-2}{x^3+4x^2-x-4}$



d) $f(x) = \frac{x^2+5x+4}{x+2}$