Section 2.2 The Factor Theorem

Building on what we learned in unit 1 and section 2.1, if we are going to analyze a polynomial function, we should figure out its roots. The key to finding the roots is to determine the factors of the equation.

The Factor Theorem

$\boldsymbol{x - b}$ is a factor of the polynomial $P(x)$ if and only if (iff) $P(b) = 0$.

Similarly, $\boldsymbol{ax - b}$ is a factor of $P(x)$ iff $P(b/a) = 0$.

The factor theorem allows you to determine the factors of a polynomial without having to divide.

Example: To determine whether $x - 3$ is a factor of $P(x) = x^3 - x^2 - 14x + 24$, we would simplify determine whether P(3) = 0.

$$P(x) = (3)^3 - (3)^2 - 14(3) + 24$$
$$P(x) = 27 - 9 - 42 + 24$$
$$P(x) = 0 \qquad \therefore (x - 3) \text{ is a factor}$$

Example: Determine whether the following binomials are factors of the polynomial, $P(x) = 2x^3 + 3x^2 - 3x - 2$.

a) $x + 2$

b) $2x - 1$

Integral Zero Theorem

If $x - b$ is a factor of a polynomial function $P(x)$ with leading coefficient 1 and the remaining coefficients that are integers, then b is a factor of the constant term of $P(x)$.

To factor a polynomial

1. Use the integral zero theorem to determine all the possible factors of the polynomials.
2. Test the factors until one of them gives you P(b) = 0.
3. Divide the polynomial by the factor you found using long division or synthetic division.
4. Repeat this process with the polynomial that remains OR use factoring by grouping. If the remaining polynomial is a quadratic use the traditional Product and Sum methods to finish factoring.

Example: Factor $P(x) = x^3 + 2x^2 - 5x - 6$ fully.

Step 1. The leading coefficient is 1, all other coefficients are integers. We are looking at possible factors of 6.

$$\pm 1, \pm 2, \pm 3, \pm 6$$

Step 2. Use the remainder theorem to find $P(b) = 0$

Let's try $P(1) = (1)^3 + 2(1)^2 - 5(1) - 6$

$P(1) = -8$ $\therefore$ $(x - 1)$ is not a factor

Let's try $P(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6$

$P(-1) = 0$ $\therefore$ $(x + 1)$ is a factor

Step 3. Divide out the factor and see what is left

$$\begin{array}{r|l}
 & x^2 + x - 6 \\
\hline
x + 1 & x^3 + 2x^2 - 5x - 6 \\
 & \underline{x^3 + x^2} \\
 & \quad x^2 - 5x \\
 & \quad \underline{x^2 + x} \\
 & \qquad -6x - 6 \\
 & \qquad \underline{-6x - 6} \\
 & \qquad\qquad 0
\end{array}$$

Step 4. Factor $x^2 + x - 6 = (x + 3)(x - 2)$

So $P(x) = x^3 + 2x^2 - 5x - 6 = (x + 1)(x + 3)(x - 2)$

Example: Factor $P(x) = x^4 + 3x^3 - 7x^2 - 27x - 18$ fully, and make a sketch of the function.

Not all polynomial functions have leading coefficients that are equal to 1. For these cases we use the Rational Zero Theorem.

Rational Zero Theorem

If $P(x)$ is a polynomial function with a leading term that is not equal to 1, but with integer coefficients and x = b/a is a zero of $P(x)$, where a and b are integers and a ≠ 0. Then

- **b** is a factor of the constant term of $P(x)$
- **a** is a factor of the leading coefficient of $P(x)$
- **ax - b** is a factor of P(x)

Example: Factor $P(x) = 3x^3 + 2x^2 - 7x + 2$ fully.

Step 1. The leading coefficient (**a**) is 3, all other coefficients are integers. The constant coefficient (**b**) is 2

$$\left.\begin{array}{l}\text{Factors of } \mathbf{b} = \pm 1, \pm 2 \\ \text{Factors of } \mathbf{a} = \pm 1, \pm 3\end{array}\right\} \text{Recall we need to test } P\left(\frac{b}{a}\right)$$

Step 2. Use the remainder theorem to find $P(b/a) = 0$

Let's try $P(1/1) = 3(1)^3 + 2(1)^2 - 7(1) + 2$

$P(1/1) = 0$ $\therefore$ $(1x - 1)$ is a factor

Step 3. Divide out the factor and see what is left

$$\begin{array}{r|l} & 3x^2 + 5x - 2 \\ \hline x - 1 & 3x^3 + 2x^2 - 7x + 2 \\ & \underline{3x^3 - 3x^2} \\ & \quad 5x^2 - 7x \\ & \quad \underline{5x^2 - 5x} \\ & \qquad -2x + 2 \\ & \qquad \underline{-2x + 2} \\ & \qquad\qquad 0 \end{array}$$

Step 4. Factor $3x^2 + 5x - 2 = (3x - 1)(x + 2)$

So $P(x) = 3x^3 + 2x^2 - 7x + 2 = (x - 1)(3x - 1)(x + 2)$

Example: Factor $P(x) = 4x^5 + 16x^4 + 3x^3 - 28x^2 - x + 6$ fully

Cheats for Factoring

Factoring a Difference of Cubes

$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

Factoring an Addition of Cubes

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

Example: Factor the following.

a) $x^3 - 8$

b) $125x^6 - 8$

c) $x^3 + 216$

d) $27x^3 + 64y^6$

Synthetic Division (an alternate approach to long division)

Divide $P(x) = x^3 + 2x^2 - 5x - 6$ by $(x - 2)$

Step 1. Setup the division chart for synthetic division

-2	1	2	-5	-6
-				
x				

i) List coefficients of the dividend in the first row
ii) To the left write the -2 from the factor (x – 2)
iii) Below the -2 place a (-) to represent the subtraction that will take place.
iv) Place an x below the horizontal axis to indicate multiplication of the divisor and the terms of the quotient.

Step 2. Carry out the operation

-2	1	2	-5	-6
-		-2	-8	-6
x	1	4	3	0

No Remainder

v) Bring down the leading coefficient
vi) Multiply the -2 with the 1 to get -2, write it underneath the second term
vii) Subtract the numbers in the second column and place the answer underneath
viii) Multiply the -2 with the four to get -8, write it underneath the third term
ix) Subtract the numbers in the third column and place the answer underneath
x) Multiply the -2 with the 3 to get -6, write it underneath the last term
xi) Subtract the numbers in the last column and place the answer underneath

$$P(x) = x^3 + 2x^2 - 5x - 6 = (x - 2)(1x^2 + 4x + 3)$$