Section 1.2 Characteristics of Polynomial Functions

In section 1.1 we explored Power Functions, a single piece of a polynomial function. This modelling method works perfectly for simple real world problems such as:

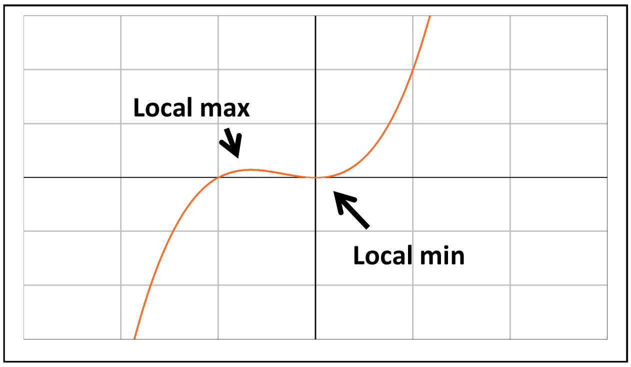
- Area Square → $A(x) = x^2$
- Volume Cube → $V(x) = x^3$
- Area Circle → $A(r) = \pi r^2$
- Volume Sphere → $V(r) = \frac{4}{3}\pi r^3$

But the more complex the situation, the more complex the function required. For example, a patient's response time to certain medication is modelled using a slightly more complex polynomial function $r(d) = -0.7d^3 + d^2$ where $r(d)$ is the reaction time in seconds, and d is the dosage of medication administered.

When combining power functions into a single polynomial function, there are a few new features we like to look for, such as

Local Minimum and Maximum Points:

Let's look at the graph of the polynomial function defined by $f(x) = x^3 + x^2$

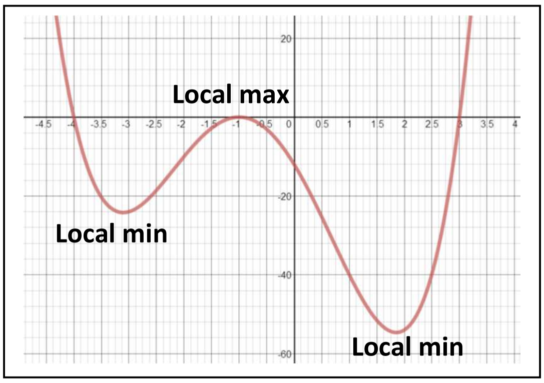


Looks more like $f(x) = x^3$ than $f(x) = x^2$, but there is a small change we now have "bumps" on the graph

In general, polynomial function graphs consist of a smooth curve with a series of hills and valleys. The hills and valleys are called **turning points**. Each turning point corresponds to a **local maximum** or **local minimum** point.

Let's look at a more complex polynomial function defined by

$$f(x) = x^4 + 3x^3 - 9x^2 - 23x - 12$$



** **The maximum possible number of local min/max points is one less than the degree of the polynomial**.**

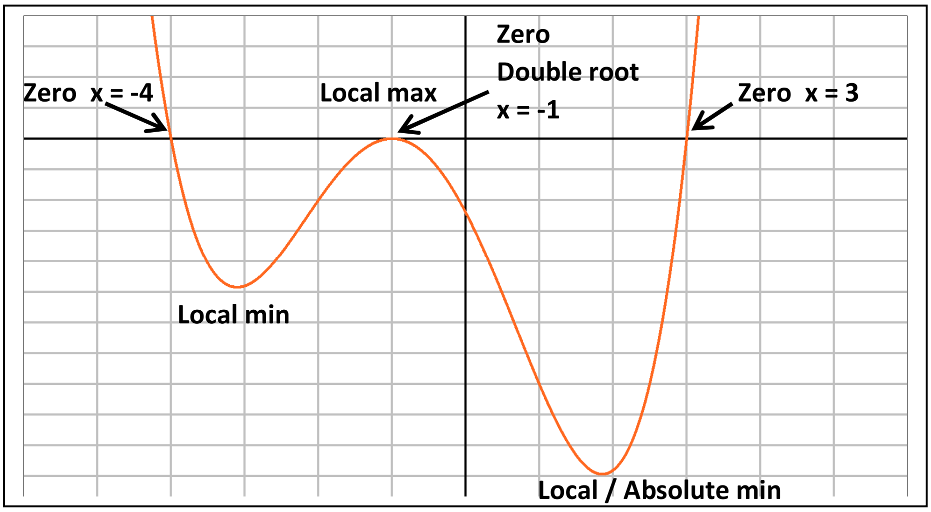
Example: The polynomial above has degree 4 and has two local minimums and one local maximum for a total of three. This is the maximum possible number of local minimum and maximum points for a polynomial of this degree.

Zeros (or x-intercepts) of polynomial functions:

A zero of a polynomial function is an x-value for which y = 0. At these x-values, its graph intersects or touches the x-axis.

** **The *maximum* number of zeros of any polynomial is the same as its degree, there may be less depending on the nature of the function and the possibility of repeated roots****

Example: The polynomial function $f(x) = x^4 + 3x^3 - 9x^2 - 23x - 12$, is shown below and only has only three zeros, not four. This is one less than the maximum of four zeros that a polynomial of degree four can have. This polynomial intersects the x-axis at x = -4 and 3, but only touches the x-axis at x = -1.



Finite Differences: (used to find leading terms and determine degree from a table of values**)**

Example:

Recall for linear functions $f(x) = 3x + 2$ we could make a table of values

x	y	1st Diff
0	2	
		3
1	5	
		3
2	8	
		3
3	11	
		3
4	14	
		3
5	17	

First Difference is constant, so degree is equal to 1 and leading coefficient is 3

Example:

Recall for quadratic functions $f(x) = 3x^2 + 2x + 1$ we could make a table of values

x	y	1st Diff	2nd Diff
0	1		
		5	
1	6		6
		11	
2	17		6
		17	
3	34		6
		23	
4	57		6
		29	
5	86		

Second Difference is constant, so degree is equal to 2 but the leading coefficient is not 6 it should be 3. So how do we account for this?

For a polynomial of degree n, where n is a positive integer, the nth differences

- are constant (equal)
- have the same sign as the leading coefficient
- are equal to a(n!), where a is the leading coefficient

Factorial (!) means: n! = n(n - 1)(n - 2)(n - 3) ... (2)(1)

5! = 5(4)(3)(2)(1)
= 120

So for our example above the second difference is constant, so degree is equal to 2 but the leading coefficient is a(n!).

6 = a(2!) because n = 2 (2^{nd} difference is where we found the constant value)
6 = a(2)(1)
6 = 2a
3 = a

Example: Each table of values represents a polynomial function. Use finite differences to determine

i) the degree of the polynomial function
ii) the sign of the leading coefficient
iii) the value of the leading coefficient

a)

x	y
0	4
1	-1
2	-12
3	-29
4	-52
5	-81
6	-116
7	-157
8	-204

b)

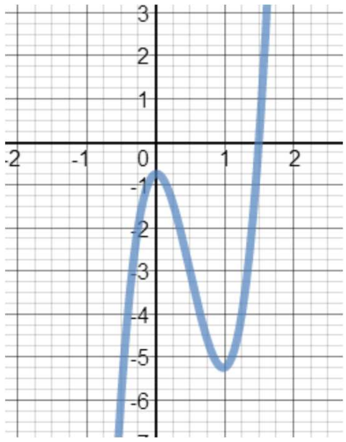
x	y
0	1
1	5
2	14
3	30
4	55
5	91
6	140
7	204
8	285

Key Features of Graphs of Polynomial Functions with Odd Degree

- Odd-degree polynomials have at least one zero, up to a maximum of n x-intercepts, where n is the degree of the function.
- The domain is {x ∈ R} and the range is {y ∈ R }.
- They have no absolute maximum point and no absolute minimum point.
- They may have point symmetry.

Positive Leading Coefficient

- Graph extends from quadrant 3 to quadrant 1.
  OR "as x→ $-\infty$ y→ $-\infty$" and "as x→ ∞, y→ ∞"



Negative Leading Coefficient

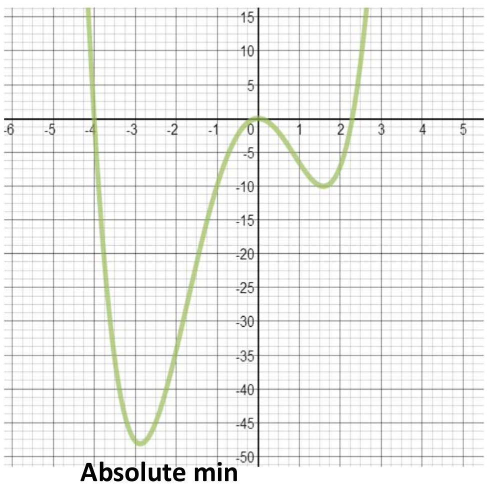
- Graph extends from quadrant 2 to quadrant 4.
  OR "as x→$-\infty$, y→∞" and "as x→∞, y→$-\infty$"

Key Features of Graphs of Polynomial Functions with Even Degree

- Even-degree polynomials may have no zeros, up to a maximum of n x-intercepts, where n is the degree of the function.
- The domain is {x ∈ R }.
- They may have line symmetry.

Positive Leading Coefficient

- Graph extends from quadrant 2 to quadrant 1.
  OR "as x→$-\infty$, y→∞" and "as x→∞, y→∞"
- The range is {y∈ R | y ≥ a}, where a is the absolute minimum value of the function.
- It will have at least one minimum point.



Negative Leading Coefficient

- Graph extends from quadrant 3 to quadrant 4.
  OR "as x→$-\infty$, y→$-\infty$" and "as x→∞, y→$-\infty$"
- The range is {y ∈ R| y ≤ a}, where a is the absolute maximum value of the function.
- It will have at least one maximum point.

Example: Determine the key features of the graph of each polynomial. Use these key features to match each function with its graph. State the number of local maximum and minimum points for the graph of each function.

a) $f(x) = 2x^3 - 4x^2 + x + 1$

b) $f(x) = -x^4 + 10x^2 + 5x - 4$

c) $f(x) = -2x^5 + 5x^3 - x$

d) $f(x) = x^6 - 16x^2 + 3$