MHF4U Unit 3 Rational Functions

Section	Pages	Questions
Prereq Skills	146-147	#1, 2, 3bf, 4ac, 6, 7ace, 8cdef, 9bf, 10abe
3.1	153-155	#1ab, 2, 3, 5ad, 6ac, 7cdf, 8, 9, 14*
3.2	164-167	#1ac, 2, 3ab, 4ab, 5acde, 8dgh, 9, 11, 14*, 16*
3.3	174-175	#1ace, 2bdf, 3ace, 5, 6bcde, 7, 8, 9, 10bc
3.4	183-185	#1, 2, 4abcf, 5bd, 9cde, 10ac, 11
3.5	189-191	#2, 3, 4, 6, 8cd, 9(determine the oblique asymptotes, only graph a)
Review	192-193 194-195	#1, 2, 3ad, 4, 5acd, 6, 7, 8, 9bcd, 10, 11, 12a, 13, 15, 16 #1-5

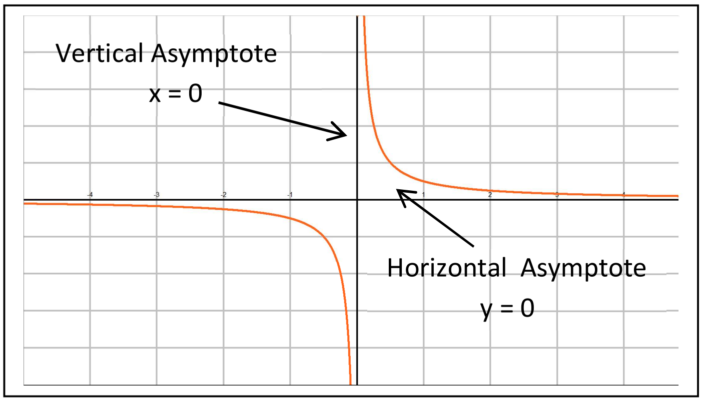
Note: Questions with an asterisk* are extra questions that are optional for the daily homework. However, they are potential "extended-type" questions that could be included on a unit test.

Section 3.0 Prerequisite Skills

Review of Reciprocal Functions

The graph of $f(x) = \frac{1}{x}$ is a graph with two asymptotes; one vertical, one horizontal.

An **asymptote** is a line that a curve approaches but never reaches.



Transformations

<u>Horizontal</u>:

If **x is multiplied by a value (k)**, it is a **horizontal stretch or compression by a factor of 1/k**. And if **k is negative**, it is a **reflection in the y-axis** (a horizontal reflection). $f(x) = \frac{1}{kx}$

If a **value (d) is added to x**, it is a **horizontal translation of d units**. $f(x) = \frac{1}{x-d}$

If both transformations are applied together, the k-value must be factored out in order to determine the correct horizontal translation. $f(x) = \frac{1}{k(x-d)}$

<u>Vertical</u>:

If the **function is multiplied by a value (a)**, it is a **vertical stretch or compression by a factor of a**. And if **a is negative**, it is a **reflection in the x-axis** (a vertical reflection). $f(x) = \frac{a}{x}$

If a **value (c) is added to the function**, it is a **vertical translation of c units**. $f(x) = \frac{1}{x} + c$

Remember that a stretch, compression, or reflection must be applied before a translation.
** R.S.T **

Domain and Range

The domain and range of a reciprocal linear function are all real numbers for x and y except for the x and y values along the two asymptotes.

For example, look at the graph of $\mathrm{f(x)} = \frac{1}{\mathrm{x}}$.
The domain is $\{\mathrm{x} \in \mathrm{R}, \mathrm{x} \neq 0\}$ and the range is $\{\mathrm{y} \in \mathrm{R}, \mathrm{y} \neq 0\}$

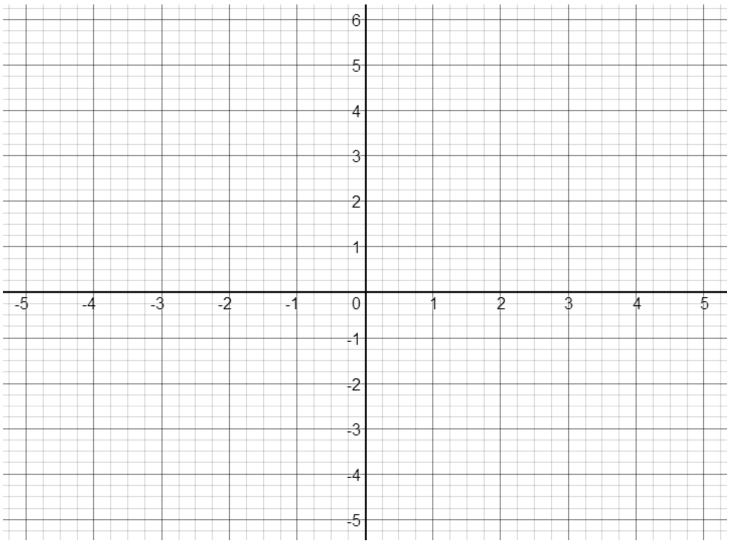
Examples: Consider the reciprocal functions below. Write an equation for the vertical and horizontal asymptotes. Use transformations to sketch the graph. Write the domain and range of the function.

a) $f(x) = \frac{1}{x+5} + 1$

b) $f(x) = \frac{-1}{2x-8}$

c) $f(x) = \frac{2}{-x-3} - 1.5$

Example: Graph $f(x) = \frac{-1}{x}$ and $f(x) = \frac{1}{-x}$ on the same grid. Compare the graphs.



Section 3.1 Reciprocal of a Linear Function

The reciprocal of a linear function has the form $f(x) = \frac{a}{kx-d} + c = \frac{1}{k\left(x-\frac{d}{k}\right)} + c$

The **restriction on the domain** of a reciprocal linear function can be determined by finding the value of x that makes the **denominator equal to zero**, that is $x = \frac{d}{k}$. Therefore, the domain of a reciprocal linear function is $\{x \in R, x \neq d/k\}$

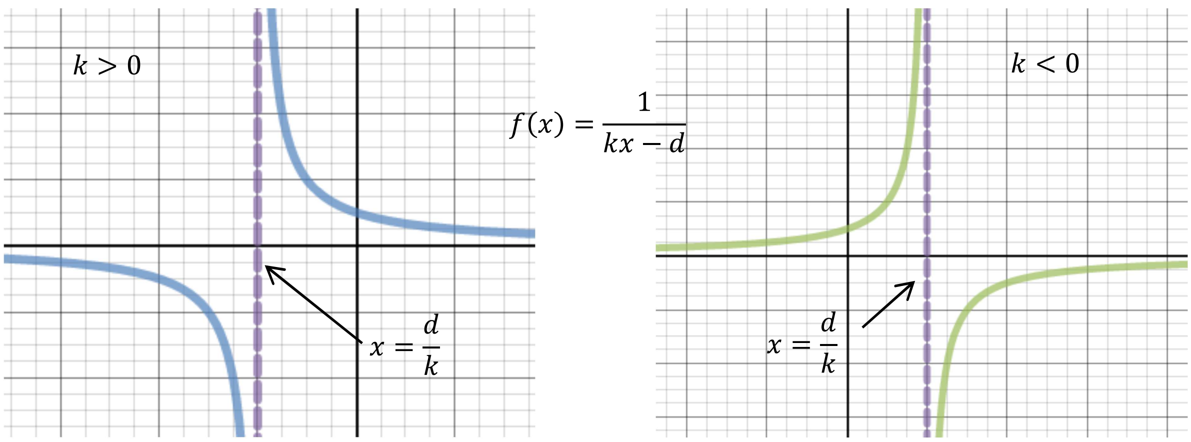
Asymptotes

The **vertical asymptote** of a reciprocal linear function occurs when x =d/k.

$x \to x^+$ means "as x approaches a from the right"

$x \to x^-$ means "as x approaches a from the left"

The **horizontal asymptote** of a reciprocal linear function of the form $f(x) = \frac{1}{kx-d} + c$ has equation y = c.



If $k > 0$, the left branch of a reciprocal linear function has a negative, decreasing slope, and the right branch has a negative, increasing slope.

If $k < 0$, the left branch of a reciprocal linear function has a positive, increasing slope, and the right branch has a positive, decreasing slope.

Example: Consider the function $f(x) = \frac{1}{x+2}$

a) State the domain

b) Make a sketch of the function

c) Describe the behaviour of the function near the vertical asymptote.

$\therefore As\ x \to\ 2^- f(x) \to$, $As\ x \to\ 2^+ f(x) \to$

d) Describe the end behaviour (as x approaches negative and positive infinity)

$\therefore As\ x \to\ -\infty\ f(x) \to$, $As\ x \to\ +\infty\ f(x) \to$

e) State the Range

f) Describe the intervals where the slope is increasing and the intervals where the slope is decreasing in the two branches of the rational function.

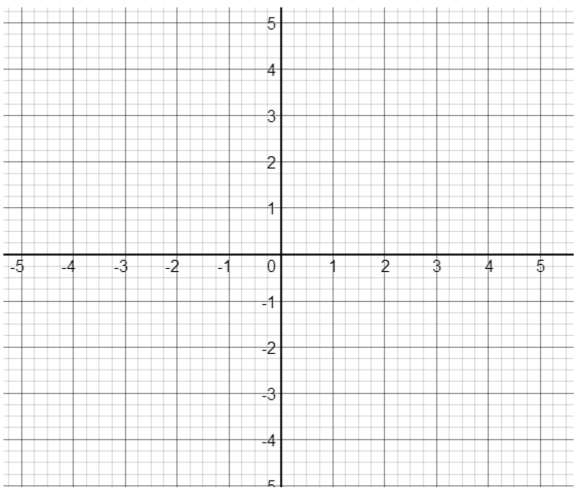
Example: Determine the x-intercepts and y-intercepts of the function $g(x) = \frac{3}{x+4}$

Example: Determine the equation in the form $f(x) = \frac{1}{kx-d}$ for the function with a vertical asymptote at x = -2 and a y-intercept at -1/10.

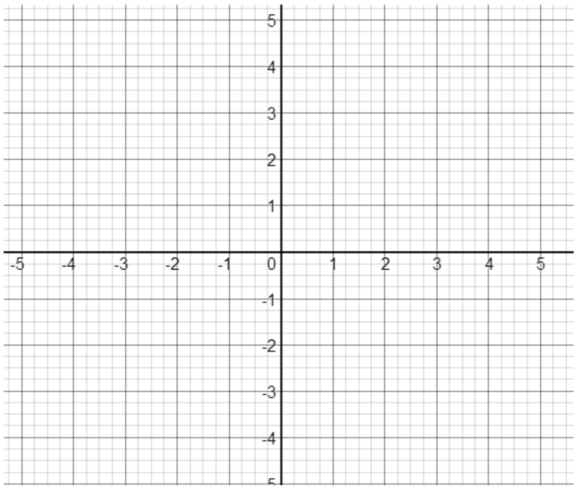
Example: For each reciprocal function

i) write an equation to represent the vertical asymptote
ii) write an equation to represent the horizontal asymptote
iii) determine the x-and y-intercepts
iv) state the domain and range
v) sketch a graph
vi) describe the intervals where the slope is increasing and where it is decreasing

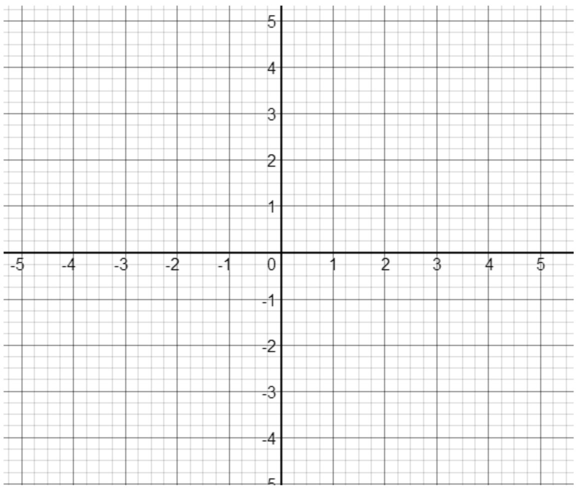
a) $f(x) = \frac{3}{x-2}$



b) $g(x) = -\frac{1}{2x+5}$



c) $h(x) = \frac{2}{1-x}$



Section 3.2 Reciprocal of a Quadratic Function

Rational functions can be analyzed using the following key features:

- Asymptotes
- Intercepts
- Slope (positive or negative, increasing or decreasing)
- Domain and Range
- Positive and Negative Intervals

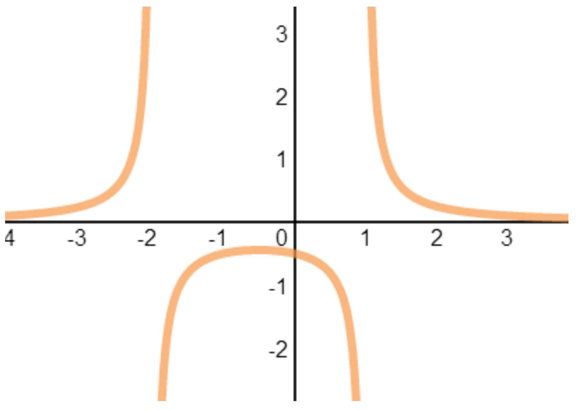
To find the domain of a quadratic function,

1. Factor the denominator, set each factor = 0 and solve each one
(ie. Determine the roots of the quadratic denominator)

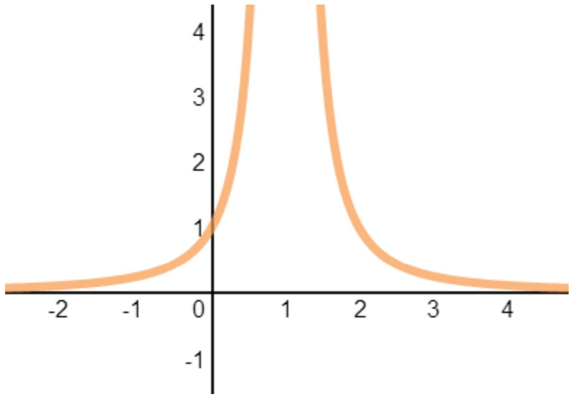
2. The solutions are the restrictions of the function and hence the values of x for which the function is not defined.

The equations of the vertical asymptotes are the restrictions for x with an = instead of a ≠.

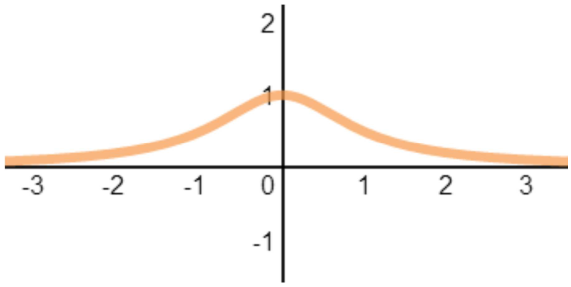
Reciprocals of quadratic functions with **two zeros** have three parts, with the middle one being a parabola. Therefore, the parabola part has a maximum or minimum point. This point is equidistant from the two vertical asymptotes and it is the x-coordinate of the vertex of the parabolic branch.



Reciprocals of quadratic functions with **only one zero** have two parts with only one vertical asymptote.



Reciprocals of quadratic functions with no real roots have only one part with no vertical asymptote.



All reciprocals of quadratic functions have a horizontal asymptote in the x-axis, at y = 0.

The behaviour near asymptotes is similar to that of reciprocals of linear functions.

All of the behaviours listed above can be predicted by analyzing the roots of the quadratic relation in the denominator.

Example: Consider the function $f(x) = \frac{2}{x^2-4}$

a) Determine the domain.

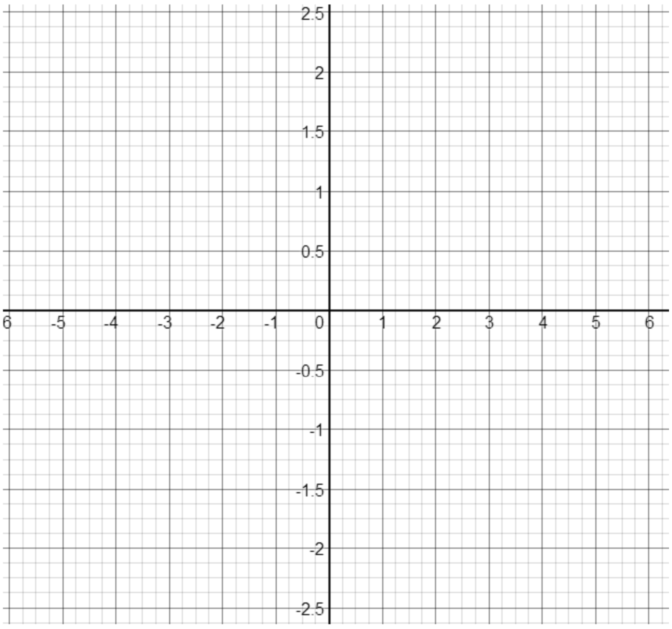
b) State the equation of the asymptotes.

c) Describe the behaviour of the function near the asymptotes and the end behaviour.

d) Determine the x and y intercepts

e) Determine the max/min point of the parabolic branch of the function.

f) Sketch the graph of the function.



g) State the range.

Example: Consider the function $f(x) = \frac{-10}{x^2-4x-5}$

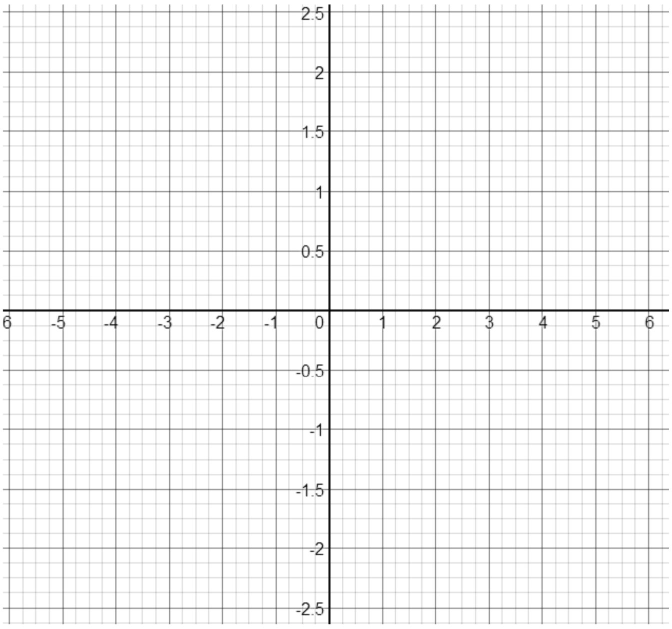
a) Determine the domain.

b) State the equations of the asymptotes.

c) Determine the intercepts.

d) Determine the min/max point of the parabolic branch of the function.

e) Sketch a graph of the function and label all important points.



f) State the range.

g) Complete the table below to summarize the intervals of increase and decrease.
Use the values of x at the asymptotes and vertex for your intervals.

Interval					
Sign of f(x)					
Sign of Slope Function is increasing (m) or decreasing (-m)					
Change in Slope (slope is increasing +) (slope is decreasing -)					

Example: For the following functions,

a) Determine the domain.
b) State the equation(s) of the asymptote(s).
c) Determine the intercepts.
d) Sketch a graph.
e) State the range.

i) $f(x) = \frac{5}{x^2+9}$

ii) $g(x) = -\frac{4}{x^2+2x+1}$

Example: Each function described below is the reciprocal of a quadratic function. Write an equation to represent each function.

a) The horizontal asymptote is y = 0. The vertical asymptotes are x = -1 and x = 3. And f(x) < 0 for the intervals x < -1 and x > 3.

b) The horizontal asymptote is y = 0. There is no vertical asymptote. Domain is $x \in R$. The maximum point is (0, 1/5).

Section 3.3 Rational Functions of the Form $f(x) = \frac{ax+b}{cx+d}$

In this section you will look at polynomial functions in which both the numerator and denominator are linear expressions.

Because there is a variable in both the numerator and denominator, there are effects on both the vertical and horizontal asymptotes and as a result the domain and range.

A rational function of the form $f(x) = \frac{ax+b}{cx+d}$ has the following key features:

- The equation of the vertical asymptote can be found by setting the denominator equal to zero and solving for x, provided the numerator does not have the same zero.

- The equation of the horizontal asymptote can be found by dividing each term in both the numerator and the denominator by x and investigating the behaviour of the function as $x \to \pm\infty$.

- The **b** constant acts to stretch the curve, but has no effect on the asymptotes, domain, or range.

- The **d** constant shifts the vertical asymptote.

- The two branches of the graph of the function are equidistant from the point of intersection of the vertical and horizontal asymptotes.

Example: Consider the function $f(x) = \frac{x+4}{x-2}$

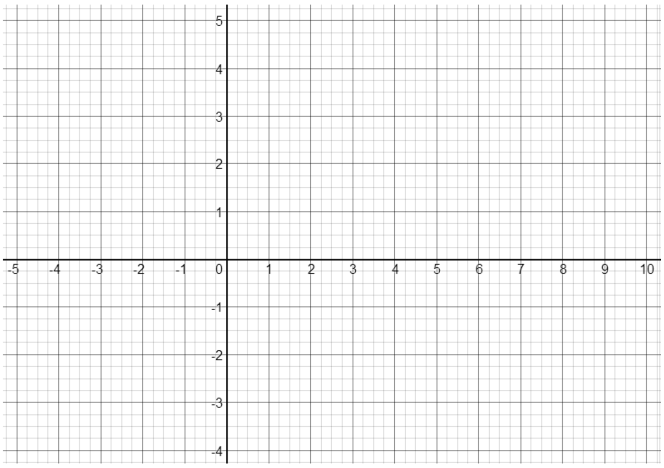
a) Determine the equation of the vertical asymptote.

b) Determine the equation of the horizontal asymptote.
(*divide each term by x and simplify*)

c) Determine the x-and y-intercepts.

d) State the domain and range.

e) Sketch a graph of the function and label all important points.



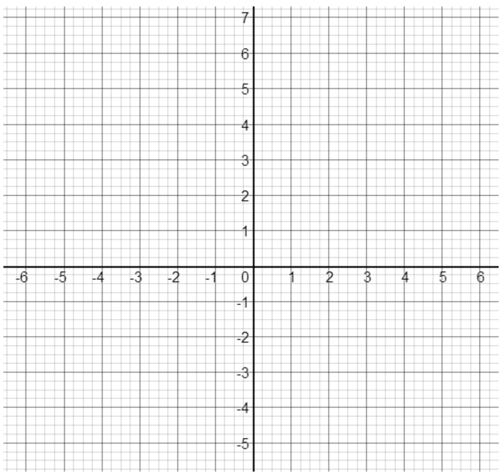
f) Complete a table to summarize the intervals of increase and decrease.
(*Include the vertical asymptotes and the x-intercepts.*)

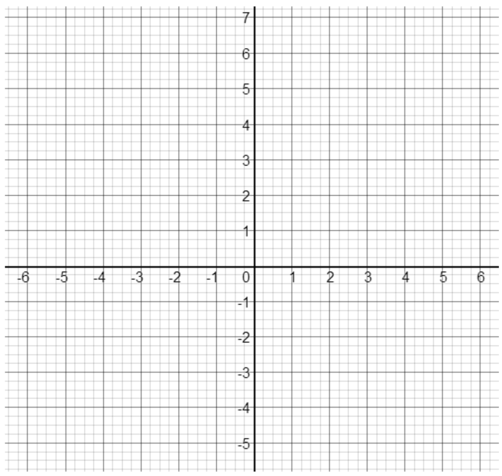
Interval				
Sign of $f(x)$				
Sign of Slope				

Examples: Compare the effects of the functions by graphing all 3 functions. Find the asymptotes and intercepts to help graph each function. State the domain and range of each function.

a) $f(x) = \frac{x-1}{2x+3}$

b) $g(x) = \frac{x-2}{2x+3}$



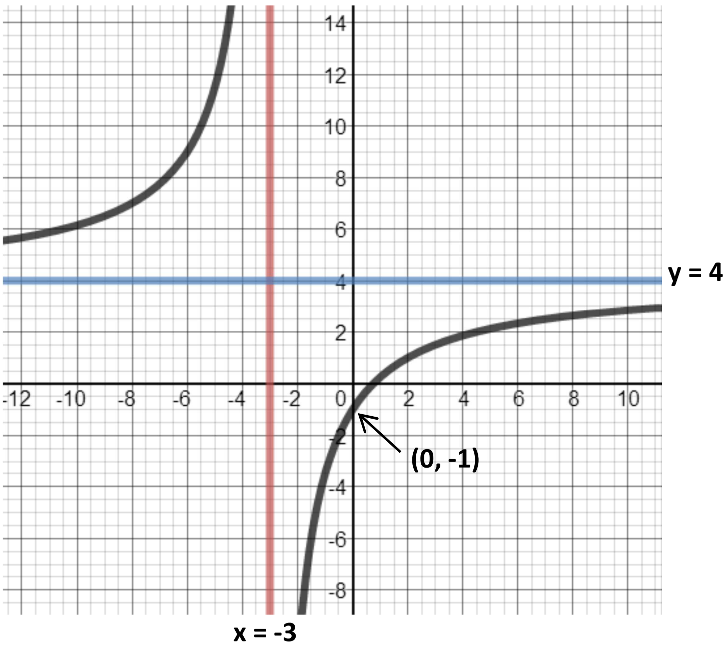


Example: Write an equation for a rational function whose graph has all of the indicated features.

→ x-intercept of $\frac{4}{7}$ → y-intercept of -2

→ horizontal asymptote at $y = \frac{7}{3}$ → vertical asymptote at $x = -\frac{2}{3}$

Example: Write an equation for the rational function shown on the graph below.



Section 3.4 Solve Rational Equations and Inequalities

To solve rational equations algebraically

- Factor the expressions in the numerator and denominator to find asymptotes and restrictions.
- Multiply both sides by the factored denominators, and simplify to obtain a polynomial equation. Then, solve using techniques you learned in unit 2.

Example: Solve algebraically. Check your solution(s) for a) and b).

a) $\frac{4}{3x-5} = 4$

b) $\frac{3x}{3x+2} - \frac{2x}{3x-2} = 1$

c) $\frac{x-5}{x^2-3x-4} = \frac{3x+2}{x^2-1}$

For rational inequalities:

- It can often help to rewrite with the right side equal to zero. Then, use test points to determine the sign of the expression in each interval.

- If there is a restriction on the variable, you may have to consider more than one case. For example, if $\frac{a}{x-k} < b$ case 1 is x $>$ k and case 2 is x $<$ k.

- Tables and number lines can help organize intervals and provide visual clue to solutions.

- The **critical values of x** are those values where there is a vertical asymptote, or where the slope of the graph of the inequality changes sign.

Example: Solve the following inequalities. Illustrate the solutions on a number line.

a) $\frac{2}{x-5} < 10$

b) $\frac{x^2-x-2}{x^2+x-12} \geq 0$

Section 3.5 Making Connections with Rational Functions and Equations

When solving a problem, it's important to read carefully to determine whether a function is being analyzed (Finding key features) or an equation or inequality is to be solved (find a missing value).

A full analysis will involve four components:

1. Numeric (tables, ordered pairs, calculations)
2. Algebraic (formulas, solving equations)
3. Graphical
4. Verbal (descriptions)

When investigating special cases of functions, factor and reduce where possible. Indicate the restrictions on the variables in order to identify hidden discontinuities.

When investigating new types of rational functions, consider what is different about the coefficients and the degree of the polynomials in the numerator and denominator. These differences could affect the stretch factor of the curve and the equations of the asymptotes and they could cause other discontinuities.

Discontinuities are values at which a function becomes undefined. They may appear as asymptotes or as holes (or gaps).

Summary of Rational Functions:

- The quotient of two polynomial functions results in a rational function which often has one or more **discontinuities**.

- The breaks or discontinuities in a rational function occur when the function is undefined. The function is undefined at values where the denominator equals zero. As a result, these values must be restricted from the domain of the function.

- The values that must be restricted from the domain of a rational function results in key characteristics that define the shape of the graph. These characteristics include a combination of vertical asymptotes (also called infinite discontinuities) and holes (also called point discontinuities).

- The end behaviours of many rational functions are determined by either horizontal asymptotes or oblique asymptotes.

* A rational function $f(x) = \frac{P(x)}{Q(x)}$ has a **hole** at **x = a** if $\frac{P(a)}{Q(a)} = \frac{0}{0}$. This occurs when P(x) and Q(x) contain a **common factor** of **(x-a)**.

For example: $f(x) = \frac{x^2-4}{x-2}$ has the common factor of (x-2) in the numerator and the denominator. This results in a hole in the graph of f(x) at x = 2

* A rational function $f(x) = \frac{P(x)}{Q(x)}$ has a **vertical asymptote** at **x = a** if $\frac{P(a)}{Q(a)} = \frac{P(a)}{0}$.

  For example: $f(x) = \frac{x+1}{x-2}$ has a vertical asymptote at x = 2

* A rational function $f(x) = \frac{P(x)}{Q(x)}$ has a **horizontal asymptote** only when the **degree of P(x) is less than or equal to the degree of Q(x)**. The equation of the horizontal asymptote is the ratio of the leading coefficients in the numerator and denominator.

  For example: : $f(x) = \frac{2x}{x+1}$ has a horizontal asymptote at y = 2.

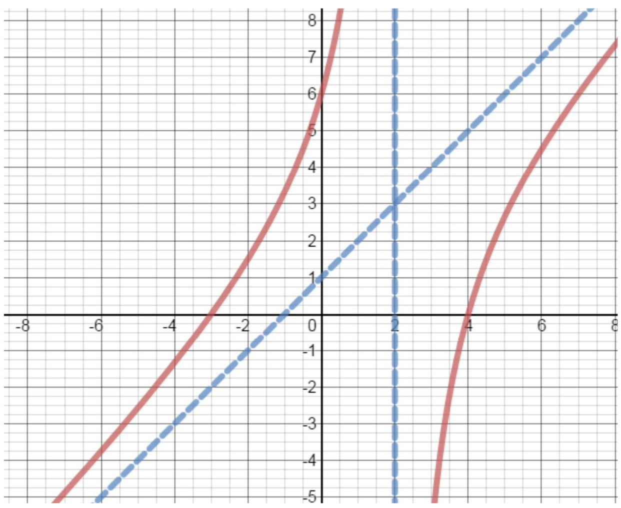
  Therefore, any reciprocal linear or reciprocal quadratic function has a horizontal asymptote at y = 0 because the degree of the numerator is 0, and the degree of the denominator is either 1 or 2.

* A rational function $f(x) = \frac{P(x)}{Q(x)}$ has an **oblique (slant) asymptote** only when the **degree of P(x) is greater than the degree of Q(x) by exactly 1**.

  To determine the equation of an oblique asymptote, divide the numerator, P(x), by the denominator, Q(x), using long division. The dividend is the equation of the oblique asymptote.

  For example, $f(x) = \frac{x^2-x-2}{x-2}$ has an oblique asymptote defined by y = x + 1

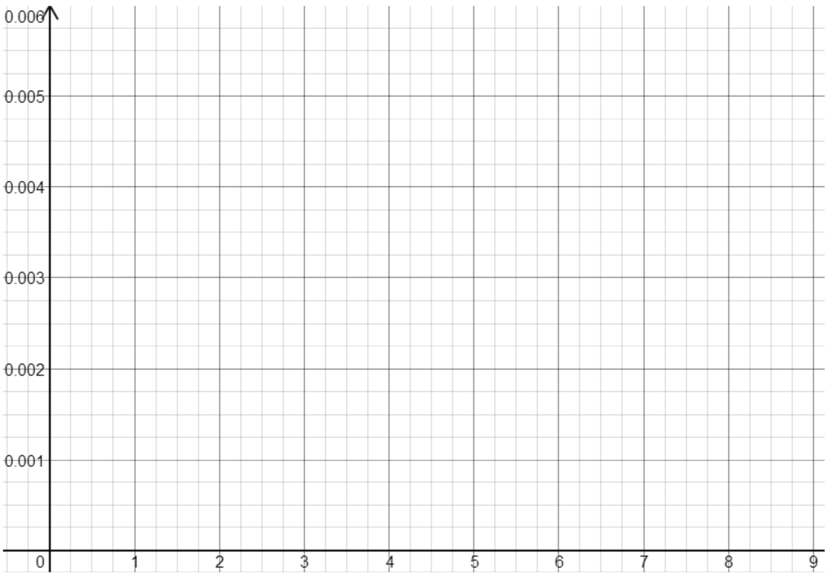
$$
\begin{array}{r|l}
 & x+1 \\
\hline
x-2 & x^2 - x - 2 \\
 & x^2 - 2x \\
\hline
 & \quad x - 2 \\
 & \quad x - 2 \\
\hline
 & \qquad 0
\end{array}
$$



Example: The intensity of sound, in watts per square meter, varies inversely as the square of the distance, in meters, from the source of the sound. So, where k is a constant. The intensity of the sound from a loudspeaker at a distance of 2 m is 0.001 W/m^2.

a) Determine the value of k, and then determine the equation of the function to represent this relationship. Write an appropriate restriction on the variable.

b) Graph the function.

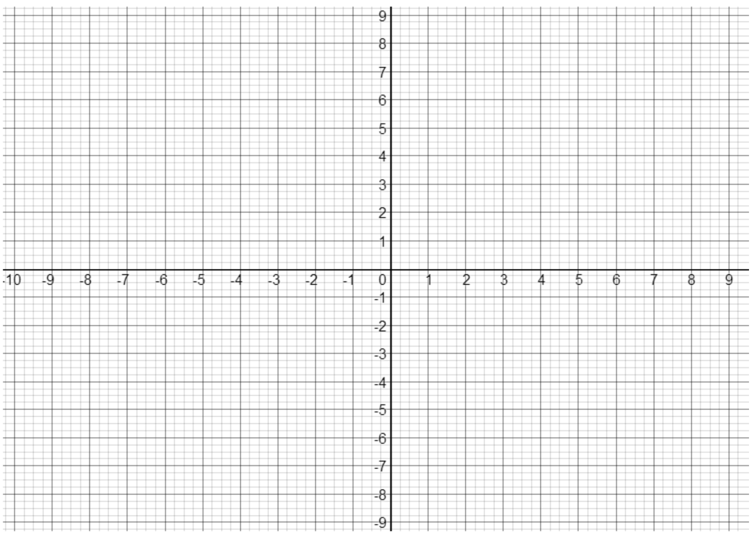


c) What is the effect of halving the distance from the source of the sound?

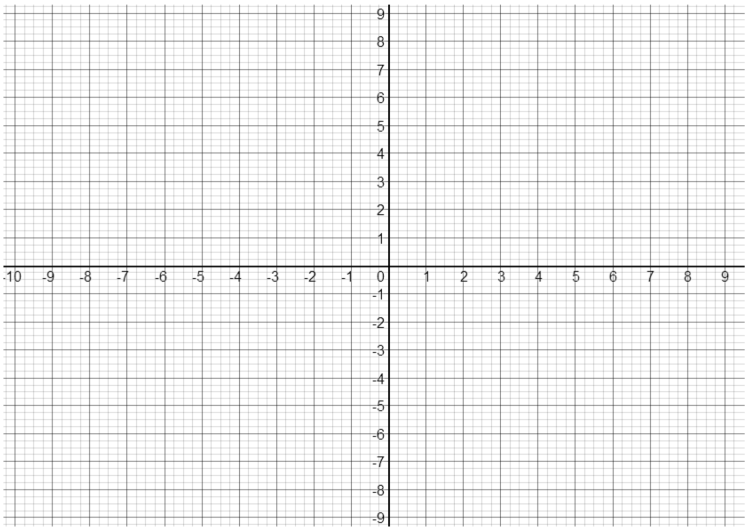
Example: In order to create a saline solution, salt water with a concentration of 40 g/L is added at a rate of 500 L/min to a tank of water that initially contained 8000 L of pure water. The resulting concentration of the solution in the tank can be modeled by the function $C(t) = \frac{40t}{160+t}$ where C is the concentration, in grams per litre, and t is the time, in minutes. In how many minutes will the saline concentration be 30 g/L?

Example: Simplify the following rational functions and state restrictions on the variable. Then sketch a graph of each simplified function and explain how these are special cases.

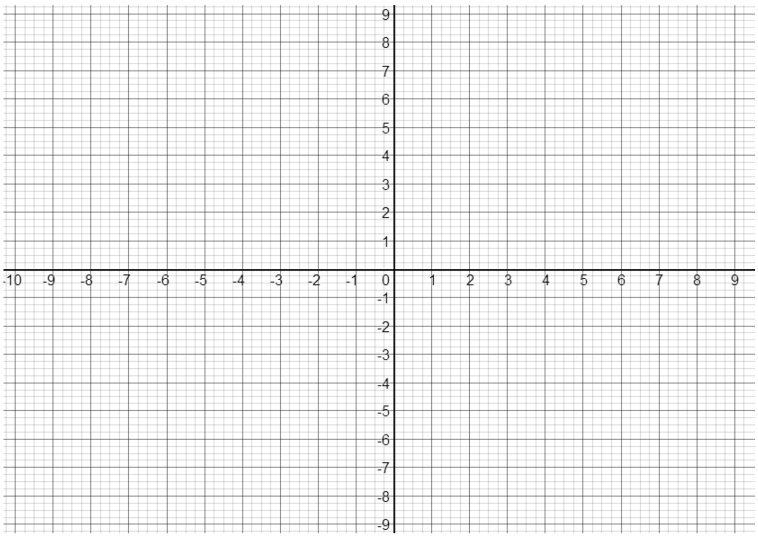
a) $f(x) = \frac{3x^2-8x+5}{x-1}$



b) $f(x) = \frac{2x^2+x-3}{2x^2+7x+6}$



c) $f(x) = \frac{2x^2-2}{x^3+4x^2-x-4}$



d) $f(x) = \frac{x^2+5x+4}{x+2}$