MHF4U Unit 2 Polynomial Equation and Inequalities

Section	Pages	Questions
Prereq Skills	82 - 83	# 1ac, 2ace, 3adf, 4, 5, 6ace, 7ac, 8ace, 9ac
2.1	91 – 93	#1, 2, 3bdf, 4ac, 5, 6, 7ab, 8c, 9ad, 10, 12, 15a, 16, 20*, 22*
2.2	102 – 103	#1bc, 2ac, 3ac, 4ace, 5ac, 6aceg, 7aceg, 9, 10, 11acf, 12a(i, iii)d(ii,iv), 13a(ii,iii)d(i,iii), 15, 17*, 18*, 20*
2.3	110 – 111	#1ace, 2ace, 3aceg, 4aceg. 5. 6aceg. 7aceg. 8ace. 9ace (inclass), 10, 14, 17*, 18*, 20*
2.4	119 – 122	#1, 2, 3, 4(don't graph), 5d, 8, 10-16
2.5	129 – 130	#1-5
2.6	138 –139	#1ace, 2, 3ac, 4ab, 5ad, 6ac, 7ac, 8, 9
Review	140 – 141 142 - 143	#1, 2, 3, 4ac, 5, 6, 7, 8, 9, 10, 12, 13, 14, 17ac, 18ab# 8abd, 13 #1-5, 11, 12

Note: Questions with an asterisk* are extra questions that are optional for the daily homework. However, they are potential "extended-type" questions that could be included on a unit test.

Section 2.1 The Remainder Theorem

In math it can be very helpful to know if one object will fit into another. Whether it can fit perfectly, or if there will be something left over.

In elementary school you were taught that to solve this problem we divided the size of the larger object by smaller object using long division. If the remainder was zero, it fit perfectly. If it was not zero, there was some amount left over.

Long Division a) $3\,\overline{)456}$ b)) $6\,\overline{)456}$

Polynomials can be divided the same way.

Ex. $x-2\,\overline{)2x^3-3x^2-3x+2}$

Focus only on the leading term in the divisor
What do we have to multiply x by to get $2x^3$?
Answer: $2x^2$. We put this on the top line, and multiply it with the divisor

$$\begin{array}{r} 2x^2 \qquad\qquad \\ x-2\,\overline{)2x^3-3x^2-3x+2} \\ \underline{2x^3-4x^2} \qquad\quad\;\; \\ x^2-3x \qquad \end{array}$$

Subtract the $2x^3-4x^2$ from the numbers above & bring the next term down (the -3x)

$$\begin{array}{r} 2x^2+x-1 \\ x-2\,\overline{)2x^3-3x^2-3x+2} \\ \underline{2x^3-4x^2} \qquad\quad\;\; \\ x^2-3x \qquad \\ \underline{x^2-2x} \qquad \\ -x+2 \\ \underline{-x+2} \\ 0 \end{array}$$

What do you have to multiply x by to get x^2?
Answer: x. We put this on the top line, and multiply it with the divisor

Subtract the x^2-2x from the numbers above & bring the next term down (the 2)

What do you have to multiply x by to get $-x$?
Answer: -1. We put this on the top line, and multiply it with the divisor

Subtract the $-x+2$ from the numbers above & your done

In this case the remainder was zero, it fit perfectly. <u>This will not always be the case.</u>

The result of the division of a polynomial function P(x) by a binomial of the form x - b is

$$\underbrace{\frac{P(x)}{x-b} = Q(x) + \frac{R}{x-b}}_{\text{Quotient Form}}$$ where Q(x) is the quotient and R is the remainder.

The corresponding statement, that can be used to check the division, is

$$P(x) = (x - b)Q(x) + R$$

So to check the result of a division, use: divisor x quotient + remainder = dividend

Examples: Divide the following polynomials. Express the result in a quotient form. Identify any restrictions on the variable. Write the corresponding statement that can be used to check the division. Verify your answer.

Dividing a Polynomial by a Binomial of the Form $x - b$

a) $(-3x^2 + 2x^3 + 8x - 12) \div (x - 1)$

Note: If the quotient (the polynomial that you are dividing) has a term in x missing, add a term by placing a zero in front of it. For example, if you are dividing $x^3 + x - 4$ by something, rewrite it as $x^3 + 0x^2 + x - 4$.

Dividing a Polynomial by a Binomial of the Form $ax - b$

b) $(4x^3 + 9x - 12) \div (2x + 1)$

Apply Long Division to Solve for Dimensions

Example: The volume, V, in cubic centimeters, of a rectangular box is given by $V(x) = x^3 + 7x^2 + 14x + 8$. Determine expressions for possible dimensions of the box if the height, h, in centimeters, is given by $x + 2$.

The Remainder Theorem

When dividing one algebraic expression by another, more often than not there will be a remainder. It is often useful to know what this remainder is and, yes, it can be calculated without going through the process of dividing as before.

The rule is:

When a polynomial function $P(x)$ is divided by $x - b$, the remainder is $P(b)$; and when it is divided by $ax - b$, the remainder is $P(b/a)$, where a and b are integers, and a $\neq$ 0.

Let's check our last two examples to see how this would work.

$P(x)$ is divided by $x - b$

a) $(-3x^2 + 2x^3 + 8x - 12) \div (x - 1)$

To determine the remainder in advance we could have just subbed $x = 1$ into the function.

$$P(b) = -3(1)^2 + 2(1)^3 + 8(1) - 12$$
$$P(b) = -3 + 2 + 8 - 12$$
$$P(b) = -5$$

$P(x)$ is divided by $ax - b$

b) $(4x^3 + 9x - 12) \div (2x + 1)$

To determine the remainder in advance we could have just subbed $x = -\frac{1}{2}$ into the function.

$$P(b/a) = 4\left(-\frac{1}{2}\right)^3 + 9\left(-\frac{1}{2}\right) - 12$$
$$P(b/a) = 4\left(-\frac{1}{8}\right) + \left(-\frac{9}{2}\right) - 12$$
$$P(b/a) = -\frac{1}{2} - \frac{9}{2} - 12$$
$$P(b/a) = -\frac{10}{2} - 12$$
$$P(b/a) = -5 - 12$$
$$P(b/a) = -17$$

Examples:

a) Use the remainder theorem to determine the remainder when $P(x) = 2x^3 + x^2 - 3x - 6$ is divided by $x + 1$. Verify your answer using long division.

b) Use the remainder theorem to determine the remainder when $P(x) = 2x^3 + x^2 - 3x - 6$ is divided by $2x - 3$. Verify your answer using long division.

c) Determine the value of k such that when $P(x) = 3x^4 + kx^3 - 7x - 10$ is divided by $x - 2$, the remainder is 8.

Section 2.2 The Factor Theorem

Building on what we learned in unit 1 and section 2.1, if we are going to analyze a polynomial function, we should figure out its roots. The key to finding the roots is to determine the factors of the equation.

The Factor Theorem

$\boldsymbol{x - b}$ is a factor of the polynomial $P(x)$ if and only if (iff) $P(b) = 0$.

Similarly, $\boldsymbol{ax - b}$ is a factor of $P(x)$ iff $P(b/a) = 0$.

The factor theorem allows you to determine the factors of a polynomial without having to divide.

Example: To determine whether $x - 3$ is a factor of $P(x) = x^3 - x^2 - 14x + 24$, we would simplify determine whether P(3) = 0.

$P(x) = (3)^3 - (3)^2 - 14(3) + 24$

$P(x) = 27 - 9 - 42 + 24$

$P(x) = 0$ $\therefore$ $(x - 3)$ is a factor

Example: Determine whether the following binomials are factors of the polynomial, $P(x) = 2x^3 + 3x^2 - 3x - 2$.

a) $x + 2$

b) $2x - 1$

Integral Zero Theorem

If $x - b$ is a factor of a polynomial function $P(x)$ with leading coefficient 1 and the remaining coefficients that are integers, then b is a factor of the constant term of $P(x)$.

To factor a polynomial

1. Use the integral zero theorem to determine all the possible factors of the polynomials.
2. Test the factors until one of them gives you P(b) = 0.
3. Divide the polynomial by the factor you found using long division or synthetic division.
4. Repeat this process with the polynomial that remains OR use factoring by grouping. If the remaining polynomial is a quadratic use the traditional Product and Sum methods to finish factoring.

Example: Factor $P(x) = x^3 + 2x^2 - 5x - 6$ fully.

Step 1. The leading coefficient is 1, all other coefficients are integers. We are looking at possible factors of 6.

$$\pm 1, \pm 2, \pm 3, \pm 6$$

Step 2. Use the remainder theorem to find $P(b) = 0$

Let's try $P(1) = (1)^3 + 2(1)^2 - 5(1) - 6$

$P(1) = -8$ $\quad\therefore$ $(x - 1)$ is not a factor

Let's try $P(-1) = (-1)^3 + 2(-1)^2 - 5(-1) - 6$

$P(-1) = 0$ $\quad\therefore$ $(x + 1)$ is a factor

Step 3. Divide out the factor and see what is left

$$\begin{array}{r|l} & x^2 + x - 6 \\ \hline x + 1 & x^3 + 2x^2 - 5x - 6 \\ & \underline{x^3 + x^2} \\ & \quad x^2 - 5x \\ & \quad \underline{x^2 + x} \\ & \qquad -6x - 6 \\ & \qquad \underline{-6x - 6} \\ & \qquad\qquad 0 \end{array}$$

Step 4. Factor $x^2 + x - 6 = (x + 3)(x - 2)$

So $P(x) = x^3 + 2x^2 - 5x - 6 = (x + 1)(x + 3)(x - 2)$

Example: Factor $P(x) = x^4 + 3x^3 - 7x^2 - 27x - 18$ fully, and make a sketch of the function.

Not all polynomial functions have leading coefficients that are equal to 1. For these cases we use the Rational Zero Theorem.

Rational Zero Theorem

If $P(x)$ is a polynomial function with a leading term that is not equal to 1, but with integer coefficients and x = b/a is a zero of $P(x)$, where a and b are integers and a ≠ 0. Then

- **b** is a factor of the constant term of $P(x)$
- **a** is a factor of the leading coefficient of $P(x)$
- **ax - b** is a factor of P(x)

Example: Factor $P(x) = 3x^3 + 2x^2 - 7x + 2$ fully.

Step 1. The leading coefficient (**a**) is 3, all other coefficients are integers. The constant coefficient (**b**) is 2

Factors of **b** = ± 1, ± 2
Factors of **a** = ± 1, ± 3 } Recall we need to test $P\left(\frac{b}{a}\right)$

Step 2. Use the remainder theorem to find $P(b/a) = 0$

Let's try $P(1/1) = 3(1)^3 + 2(1)^2 - 7(1) + 2$

$P(1/1) = 0$ ∴ $(1x - 1)$ is a factor

Step 3. Divide out the factor and see what is left

$$\begin{array}{r|l} & 3x^2 + 5x - 2 \\ \hline x - 1 & 3x^3 + 2x^2 - 7x + 2 \\ & \underline{3x^3 - 3x^2} \\ & 5x^2 - 7x \\ & \underline{5x^2 - 5x} \\ & -2x + 2 \\ & \underline{-2x + 2} \\ & 0 \end{array}$$

Step 4. Factor $3x^2 + 5x - 2 = (3x - 1)(x + 2)$

So $P(x) = 3x^3 + 2x^2 - 7x + 2 = (x - 1)(3x - 1)(x + 2)$

Example: Factor $P(x) = 4x^5 + 16x^4 + 3x^3 - 28x^2 - x + 6$ fully

Cheats for Factoring

Factoring a Difference of Cubes

$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

Factoring an Addition of Cubes

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

Example: Factor the following.

a) $x^3 - 8$

b) $125x^6 - 8$

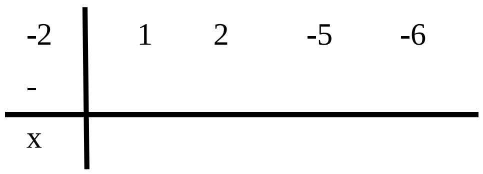
c) $x^3 + 216$

d) $27x^3 + 64y^6$

Synthetic Division (an alternate approach to long division)

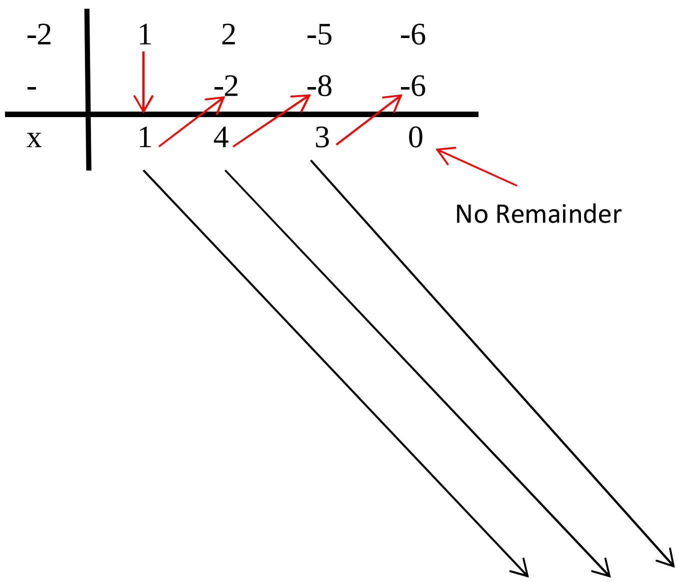
Divide $P(x) = x^3 + 2x^2 - 5x - 6$ by $(x - 2)$

Step 1. Setup the division chart for synthetic division



i) List coefficients of the dividend in the first row
ii) To the left write the -2 from the factor (x – 2)
iii) Below the -2 place a (-) to represent the subtraction that will take place.
iv) Place an x below the horizontal axis to indicate multiplication of the divisor and the terms of the quotient.

Step 2. Carry out the operation



v) Bring down the leading coefficient
vi) Multiply the -2 with the 1 to get -2, write it underneath the second term
vii) Subtract the numbers in the second column and place the answer underneath
viii) Multiply the -2 with the four to get -8, write it underneath the third term
ix) Subtract the numbers in the third column and place the answer underneath
x) Multiply the -2 with the 3 to get -6, write it underneath the last term
xi) Subtract the numbers in the last column and place the answer underneath

$P(x) = x^3 + 2x^2 - 5x - 6 = (x - 2)(1x^2 + 4x + 3)$

Section 2.3 Polynomial Equations

- The **real roots** of a polynomial equation $P(x) = 0$ correspond to the x-intercepts of the graph of the polynomial function $P(x)$.

- The x-intercepts of the graph of a polynomial function correspond to the real roots of the related polynomial equation. They can also be called the zeros of the function.

- If a polynomial equation is factorable, the roots are determined by factoring the polynomial, setting its factors equal to zero, and solving each factor.

- If a polynomial equation is not factorable, the roots can be determined from the graph using technology.

Example: Solve the following polynomials by factoring. Then use the x-intercepts and the end behaviour of the functions to sketch a graph.

a) $-x^3 + x^2 + 6x = 0$ b) $2x^3 - x^2 - 18x + 9 = 0$ c) $x^3 - 3x^2 + x = 3$

Example: Solve the following using the factor theorem. Round to 2 decimal places where necessary.

a) $2x^3 + 3x^2 - 11x = 6$

b) $x^3 + 3x^2 - 11x + 7 = 0$

Example: The volume, V, in cubic centimeters, of a block of wood that a sculptor uses to carve a wolf can be modelled by $V(x) = 9x^3 + 3x^2 + 120x$, where x represents the thickness of the block, in centimeters. What maximum thickness of wolf can be carved from a block of wood with a volume of 1332 cm^3?

Section 2.4 Families of Polynomial Functions

- A **family of functions** is a set of functions with the **same characteristics**.
- Polynomial functions with the **same zeros** are said to belong to the same family. The graphs of polynomial functions that belong to the same family have the **same x-intercepts** but **different y-intercepts** (unless zero is one of the x- intercepts).
- A family of polynomial functions with zeros a1, a2, a3,... an, can be represented by an equation of the form $f(x) = k(x - a_1)(x - a_2)(x - a_3)\ldots(x - a_n)$, where $k \in R, k \neq 0$.
- An equation for a particular member of a family of polynomial functions can be determined if a point on the graph is known.

Example: The zeros of a family of quadratic functions are -3 and 2.

a) Determine an equation for this family.

$f(x) = k(x + 3)(x - 2)$

b) Write equations for two functions that belong to this family.

$f(x) = 35(x + 3)(x - 2)$

$f(x) = -1(x + 3)(x - 2)$

c) Determine an equation for the member of the family whose graph has a y-intercept of -18.

$-18 = k(0 + 3)(0 - 2)$ x = 0 for y-intercept
$-18 = -6k$
$3 = k$

So $f(x) = 3(x + 3)(x - 2)$

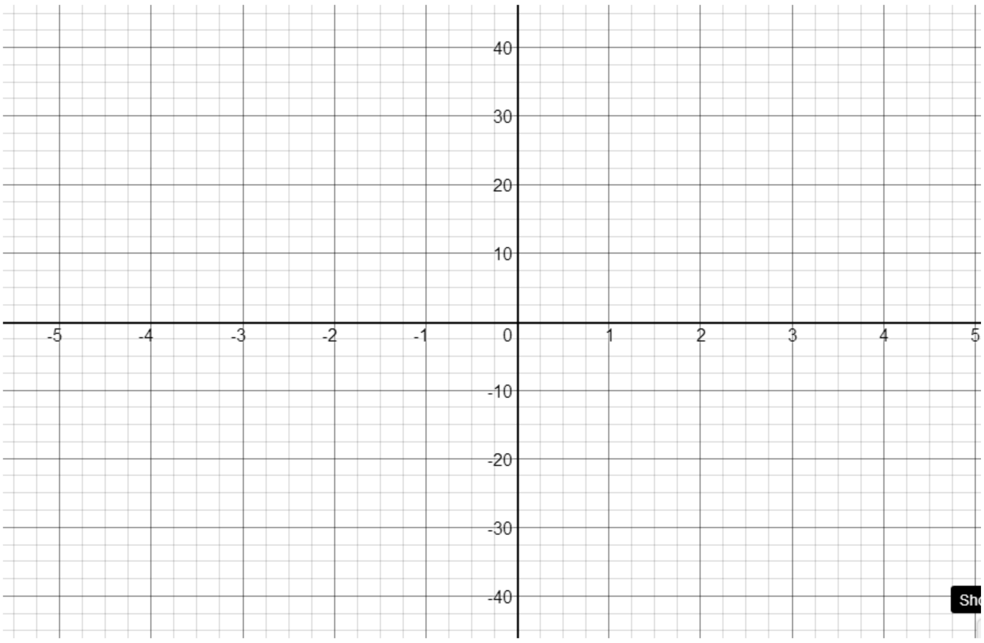
Example: The zeros of a family of cubic functions are -3, 1, and 4.

a) Determine an equation for this family.

b) Write equations for two functions that belong to this family.

c) Determine an equation for the member of the family whose graph has a y-intercept of -18.

d) Sketch graphs of the functions in parts b) and c).

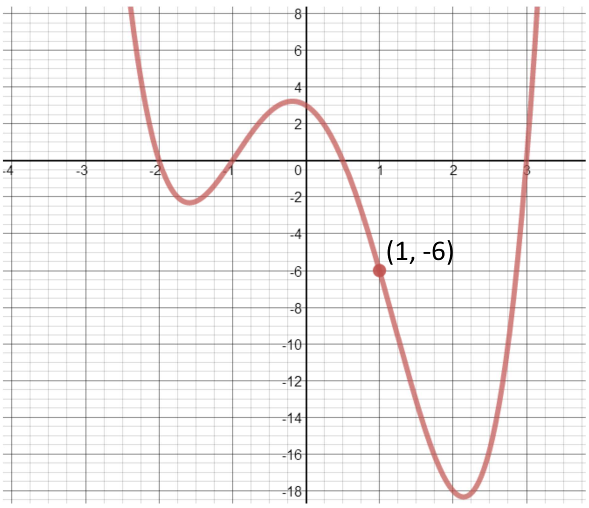


Example:

a) Determine a simplified equation for the family of quartic equations with zeros at ±1 and $2 \pm \sqrt{3}$.

b) Determine an equation for the member of the family whose graph passes through the point (2, 18).

Example: Determine an equation for the quartic function represented by this graph.



Section 2.5 Solving Polynomial Functions Using Tech.

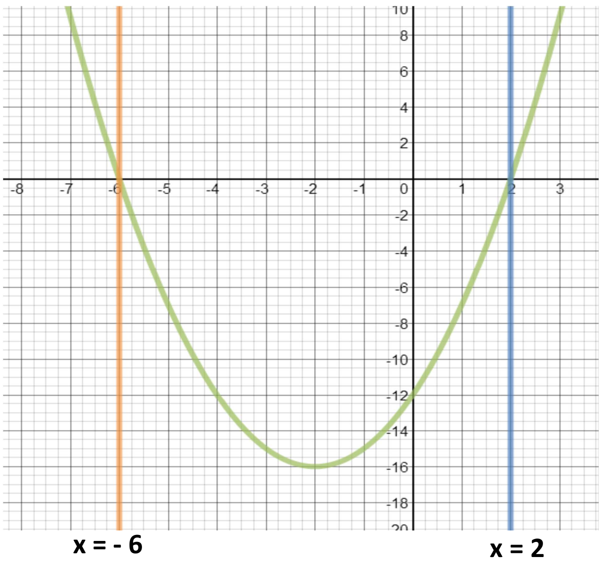
- A polynomial inequality results when the equal sign in a polynomial equation is replaced with an inequality symbol. $<$ (Less than) $>$ (Greater than)

- The real zeros of a polynomial function, or x-intercepts of the corresponding graph, divide the x-axis into intervals that can be used to solve a polynomial inequality.

- Polynomial inequalities may be solved graphically by determining the x-intercepts and then using the graph to determine the intervals that satisfy the inequality.

- A CAS (computer algebra system) on a graphing calculator may be used to solve a polynomial inequality numerically by determining the roots of the polynomial equation and then testing values in each interval to see if they make the inequality true.

Examine the graph of $f(x) = x^2 + 4x - 12$



The x-intercepts are -6 and 2. These correspond to the zeros of the function $f(x) = x^2 + 4x - 12$

By moving from left to right along the x-axis, we can make the following observations.

- The function is positive when $x < -6$ since the y-values are positive

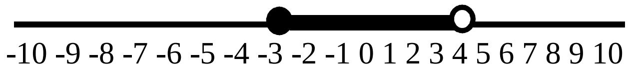
- The function is negative when $-6 < x < 2$ since the y-values are negative.

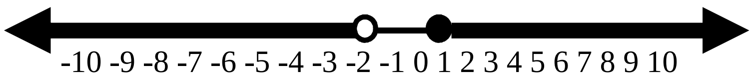
- The function is positive when $x > 2$ since the y-values are positive

The zeros -6 and 2 divide the x-axis into three intervals: $x < -6$, $-6 < x < 2$ and $x > 2$. In each interval, the function is either positive or negative. The information can be summarized in a table, as shown below.

Interval	$x < -6$	$-6 < x < 2$	$x > 2$
Sign of Function	+	•	+

Examples: Write inequalities for the values of x shown.

a) 

b) 

Example: Write intervals into which the x-axis is divided by each set of x-intercepts of a polynomial function.

a) -1/2 , 5

b) - 4, 0, 1

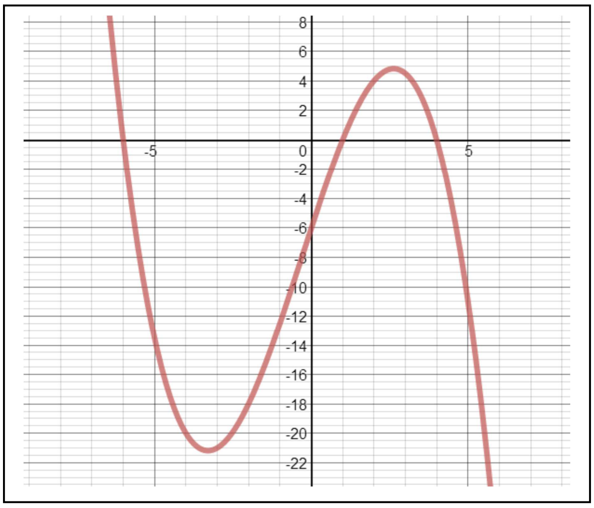
Example: Sketch a graph of a cubic polynomial function $y = f(x)$ such that

$f(x) < 0$ when $x < -3 \; or \; -1 < x < 5$, and
$f(x) > 0$ when $-3 < x < -1 \; or \; x > 5$.

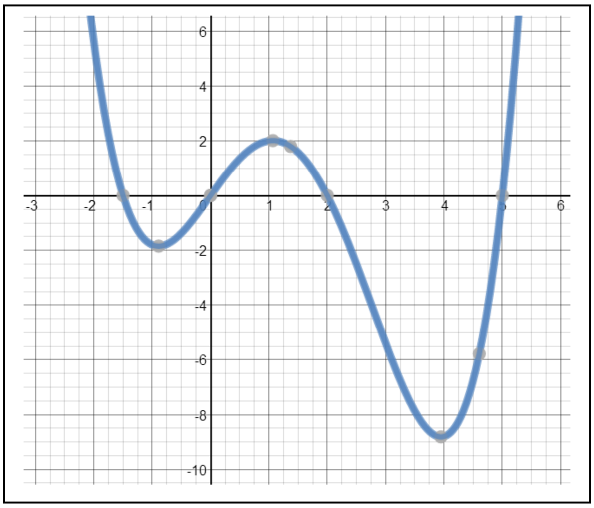
Example: For the following graphs, write the

i) x-intercepts
ii) intervals of x for which the graph is positive
iii) intervals of x for which the graph is negative

a)



b)



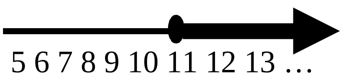
Section 2.6 Solve Factorable Polynomial Inequalities Algebraically

Solving Linear Inequalities

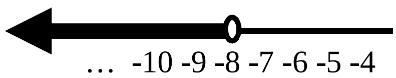
- Solve as you would a regular equation.
- Remember to flip the inequality sign when dividing/multiplying by a negative number.

Example: Solve each linear inequality and graph your solution on a number.

a) $x - 7 \geq 4$
$x \geq 4 + 7$
$x \geq 11$



b) $-4 - 2x > 12$
$-2x > 12 + 4$
$-2x > 16$
$x < -8$



Solve Polynomial Inequalities

- Factorable inequalities can be solved algebraically by factoring the polynomial, if necessary, and determining the zeros/roots of the function.

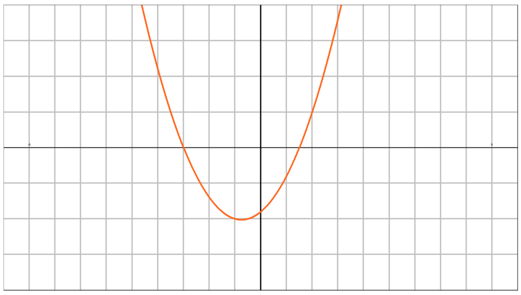
Then...

1. Consider all cases, OR
2. Use intervals and then test values in each interval

- Tables and number lines can help organize intervals to provide a visual clue to solutions.

Example: Solve the inequality using cases and intervals.

a) $(x + 3)(2x - 3) > 0$



Roots x = - 3, 3/2

The polynomial is already in factored form so we have saved a little work. This will not always be true.

We will start with the pure **Algebra** based solution.

- Step 1 – determine the number of possible cases for the inequalities

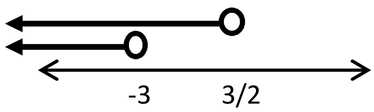
  We have two brackets being multiplied with the goal being to determine when this function will be greater than zero. i.e when is the function positive

  The product will be positive when both brackets have a positive value (case 1) or when both brackets have negative values (case 2).

  So we have two cases

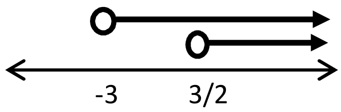
- Step 2 – Solve for both cases (determine when are both true)

Case 1	$x + 3 > 0$	$2x - 3 > 0$
	$x > -3$	$2x > 3$
		$x > 3/2$



Both will be positive for numbers less than – 3

Case 2	$x + 3 < 0$	$2x - 3 < 0$
	$x < -3$	$2x < 3$
		$x < 3/2$



Both will be negative for numbers greater than 3/2

- Step 3 Write your concluding inequality statement

  $\therefore (x + 3)(2x - 3) > 0$ when $x < -3$ & $x > 3/2$

Let's try this problem a second way using the interval method

Intervals: Start with the idea that this function has the **potential** to change from positive to negative values at the roots. We say potential because it could just touch the axis and bend back.

- We will create a table to discuss all regions for the function in space,
- We will test values in these regions in each of the factors to determine the sign of the function.

- We know the roots of this function are at x = -3, 3/2 so let us discuss the interval before -3, between -3 and 3/2, and after 3/2.

- We will still use the logic from the algebra solution, that both factors must either be positive or negative to provide a result that is > 0.

Interval	x < -3	-3 < x < 3/2	x > 3/2
Factors	Try – 4	Try 1	Try 4
(x + 3)	(-4 + 3) = -1 Sign (-)	(1 + 3) = 4 Sign (+)	(4 + 3) = 7 Sign (+)
(2x – 3)	[2(-4) – 3] = -11 Sign (-)	[2(1) – 3] = -1 Sign (-)	[2(4) – 3] = 5 Sign (+)
Result (x+3)(2x -3)	(+)	(-)	(+)

$\therefore (x + 3)(2x - 3) > 0$ when x < -3 & x > 3/2

Example: Solve the inequality using cases and intervals.

b) $-2x^3 - 6x^2 + 12x + 16 \leq 0$

Example: The price, P, in dollars, of a stock t years after 2000 can be modelled by the function $P(t) = 0.4t^3 - 4.4t^2 + 11.2t$. When will the price of the stock be more than \$36?